

Solar-System Constraints on $f(R)$ Chameleon Gravity

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Abstract

We investigate the solar-system constraint on the $f(R)$ theory of modified gravity with chameleon mechanism, where $f(R)$ represents the deviation from general relativity in the gravity action. We obtain a stringent bound to a general, non-constant deviation function $f(R)$: $-10^{-15} \lesssim df/dR < 0$ when $R/H_0^2 \sim 3 \times 10^5$, and a loose bound: $0 < R d^2f/dR^2 < 2/5$ when $R/H_0^2 \gtrsim 3 \times 10^5$, by requiring the thin-shell condition in the solar system, particularly in the atmosphere of the Earth. These bounds can be conveniently utilized to test the $f(R)$ models with given functional forms of $f(R)$ and to obtain the constraints on the parameters therein. For demonstration we apply these bounds to several widely considered $f(R)$ models.

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I. INTRODUCTION

The discovery of the cosmic acceleration at the present epoch [1, 2] indicates the existence of repulsive gravity that dominates the present universe at large scales. It may be an indication of an energy source of anti-gravity or a sign of the modification of the gravity theory. Along the latter direction the $f(R)$ theory of modified gravity [3–13] (for a review, see [14]) has been proposed as a possible explanation of the cosmic acceleration, where the gravity is described by a function of the Ricci scalar R in the action. On the other hand, the $f(R)$ theory can be treated as a simple way of modeling the possible deviation from Einstein’s general relativity (GR).

As an essence of cosmology, $f(R)$ gravity needs to pass the cosmological test [15–24] involving the observations of the cosmic expansion and the cosmic structure formation. As a gravity theory, it needs to pass the local gravity test that in general gives the most stringent constraint on modified gravity so far [15, 16, 25–28]. It has been pointed out that $f(R)$ gravity, as a special case of the scalar-tensor theory, may pass the solar-system test with the help of the “chameleon mechanism” [29, 30], meanwhile driving the late-time cosmic acceleration. In the chameleon mechanism the scalar field of the scalar-tensor theory can behave differently in different environments, depending on the ambient mass density. This feature makes it possible to have significant deviations from GR at the cosmological scales at late times and meanwhile have tiny deviations both in the solar system at present and at all scales at early times.

In this paper we investigate the solar-system constraint on the $f(R)$ gravity with the chameleon mechanism.¹ As a result, we obtain the following constraints on a general, non-constant function $f(R)$ that represents the deviation from GR in the gravity action.

$$-10^{-15} \lesssim df/dR < 0 \quad \text{when} \quad R/H_0^2 \sim 3 \times 10^5, \quad (1)$$

$$0 < R d^2 f/dR^2 < 2/5 \quad \text{when} \quad R/H_0^2 \gtrsim 3 \times 10^5. \quad (2)$$

These constraints will be derived in Sec. IV. Before that, in Sec. II we will introduce $f(R)$ gravity, the chameleon mechanism and the thin-shell condition, and in Sec. III elaborate on the thin-shell parameter in $f(R)$ gravity and obtain its relation to $f(R)$, with which the solar-system constraint on the thin-shell parameter can be transferred to that on $f(R)$.

¹ We consider the metric formalism of $f(R)$ gravity.

II. $f(R)$ GRAVITY WITH CHAMELEON MECHANISM

We consider the $f(R)$ theory of modified gravity with the action,²

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R + f(R)] + S_m(g_{\mu\nu}, \Psi_m), \quad (3)$$

where $f(R)$ is a function of the Ricci scalar R and represents the deviation from GR, κ is the gravitational constant, S_m the matter action and Ψ_m the matter field. This theory can be transformed to a scalar-tensor theory in the Einstein frame via a conformal transformation,

$$\tilde{g}_{\mu\nu} = (1 + f_R) g_{\mu\nu} = e^{-2\beta\kappa\phi} g_{\mu\nu}, \quad \beta = -1/\sqrt{6}, \quad (4)$$

where $f_R \equiv df/dR$. The resultant action in the Einstein frame is

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2\kappa^2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) - V(\phi) \right] + S_m(e^{2\beta\kappa\phi} \tilde{g}_{\mu\nu}, \Psi_m), \quad (5)$$

where

$$\phi = -\frac{1}{2\beta\kappa} \ln [1 + f_R(R)], \quad (6)$$

$$V(\phi) = \frac{R f_R(R) - f(R)}{2\kappa^2 [1 + f_R(R)]^2}, \quad (7)$$

$$\tilde{R} = R [\tilde{g}_{\mu\nu}]. \quad (8)$$

The notation about the derivatives of $f(R)$, $f_R \equiv df/dR$ and $f_{RR} \equiv d^2f/dR^2$, will be used in the remainder of this paper.

The potential V in Eq. (7) is a function of the scalar field ϕ , provided that the Ricci scalar R is a function of ϕ given by Eq. (6) with a well-defined inverse function f_R^{-1} . In the action S_E the scalar field acquires an additional coupling to matter through the Jordan-frame metric tensor $g_{\mu\nu}$ in the matter action S_m . This additional coupling is the key to the chameleon mechanism. It makes the scalar field ϕ behave differently in different environments, depending on the ambient mass density.

To study the $f(R)$ chameleon gravity with the action S_E , for simplicity we consider a spherically symmetric system, especially a sphere with different constant mass densities inside and outside the sphere. In the following we will present the field equation and the

² We use the convention, $\{-, +, +, +\}$, for the metric signature.

solution for the scalar field $\phi(r)$, where r is the physical distance from the center of the system.

For a spherically symmetric space-time with a mass density distribution $\rho^*(r)$, the field equation is

$$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = \frac{\partial V_{\text{eff}}(\phi, \rho^*)}{d\phi}, \quad (9)$$

where the effective potential

$$V_{\text{eff}}(\phi, \rho^*) \equiv V(\phi) + e^{\beta\kappa\phi} \rho^*. \quad (10)$$

The mass density ρ^* is a conserved quantity in the Einstein frame. By $\rho^* = e^{3\beta\kappa\phi} \rho$, it is related to the mass density ρ in the $f(R)$ gravity with the action S .

To satisfy the constraints from the solar-system experiments, the effective potential V_{eff} should have a minimum and should be steep around the minimum when the mass density ρ^* is equal to those in the solar system, including the regions inside the Earth ($\rho_{\oplus}^* \simeq 5.5 \text{ g/cm}^3$), in the atmosphere ($\rho_{\text{atm}}^* \sim 10^{-3} \text{ g/cm}^3$) and outside the atmosphere ($\rho_G^* \sim 10^{-24} \text{ g/cm}^3$). Let ϕ_m denote the location of the potential minimum and m the mass of ϕ , i.e.,

$$\partial_{\phi} V_{\text{eff}}(\phi_m, \rho^*) = 0, \quad (11)$$

$$m^2(\rho^*) \equiv \partial_{\phi}^2 V_{\text{eff}}(\phi_m, \rho^*). \quad (12)$$

The basic condition raised above is then recast by the requirements: (1) the existence of ϕ_m , (2) positive m^2 , and (3) large m (i.e. short Compton wavelength), when $\rho^* \sim \rho_{\oplus}^*, \rho_{\text{atm}}^*, \rho_G^*$, etc.

We consider a sphere with the radius r_s and with different constant mass densities, ρ_{in}^* and ρ_{out}^* , inside and outside the sphere. The locations of the potential minima ϕ_c and ϕ_{∞} , the masses m_{in} and m_{out} , the total mass of the sphere M_s and the Newtonian potential Φ_s at r_s (i.e. on the surface of the sphere) are defined as follows.

$$\partial_{\phi} V_{\text{eff}}(\phi_c, \rho_{\text{in}}^*) = 0, \quad \partial_{\phi} V_{\text{eff}}(\phi_{\infty}, \rho_{\text{out}}^*) = 0, \quad (13)$$

$$m_{\text{in}}^2 \equiv \partial_{\phi}^2 V_{\text{eff}}(\phi_c, \rho_{\text{in}}^*), \quad m_{\text{out}}^2 \equiv \partial_{\phi}^2 V_{\text{eff}}(\phi_{\infty}, \rho_{\text{out}}^*), \quad (14)$$

$$M_s \equiv \frac{4\pi}{3} r_s^3 \rho_{\text{in}}^*, \quad \Phi_s \equiv \frac{\kappa^2}{8\pi} \frac{M_s}{r_s}. \quad (15)$$

The ϕ field profile, as a solution of Eq. (9), is

$$\begin{cases} \phi(r < r_s) \simeq \phi_c, \\ \phi(r > r_s) \simeq -\left(\frac{\beta\kappa}{4\pi}\right)\left(\frac{3\Delta r_s}{r_s}\right)\frac{M_s e^{-m_{\text{out}}(r-r_s)}}{r} + \phi_\infty, \end{cases} \quad (16)$$

when the following thin-shell condition is satisfied.

$$0 < \frac{\Delta r_s}{r_s} \equiv \frac{\kappa(\phi_\infty - \phi_c)}{6\beta\Phi_s} \ll 1. \quad (17)$$

The thin-shell parameter $\Delta r_s/r_s$ is proportional to the ratio of two potential differences, $\kappa\phi_\infty - \kappa\phi_c$ and $\Phi_s - \Phi_\infty$ [where $\Phi_\infty \equiv \Phi(r = \infty) \equiv 0$], which respectively relate to the strength of the fifth force induced by ϕ and that of the Newtonian gravitational force. Thus, roughly speaking, the thin-shell condition requires the weakness of the fifth force compared to the Newtonian gravity.

The ϕ profile in Eq. (16) depends on the thin-shell parameter $\Delta r_s/r_s$ that is determined when the function $f(R)$ and the mass densities, ρ_{in}^* and ρ_{out}^* , are given. Accordingly, the solar-system bounds to the fifth force can constrain $\Delta r_s/r_s$ and thereby constrain $f(R)$.

III. THIN-SHELL PARAMETER IN $f(R)$ GRAVITY

Here we will derive the relation between the thin-shell parameter $\Delta r_s/r_s$ and the function $f(R)$. With this relation at hand we can obtain the constraint on general $f(R)$ from the solar-system tests of gravity which give bounds to the thin-shell parameter. In addition, the requirement of the existence of the potential minima, $m_{\text{in}}^2 > 0$ and $m_{\text{out}}^2 > 0$, also gives a basic condition of $f(R)$, for which we will derive the relation between $f(R)$ and the masses, m_{in} and m_{out} .

The thin-shell parameter is related to $f(R)$ through the locations of the potential minima, ϕ_c and ϕ_∞ , which satisfy Eq. (13). With $dV/d\phi = (dV/dR)(dR/d\phi)$ and $e^{\beta\kappa\phi} = 1/\sqrt{1+f_R}$ we have

$$\partial_\phi V_{\text{eff}}(\phi, \rho^*) = \frac{dV}{d\phi} + \beta\kappa e^{\beta\kappa\phi} \rho^* \quad (18)$$

$$= \frac{R + 2f - Rf_R}{\sqrt{6\kappa}(1+f_R)^2} + \frac{\beta\kappa\rho^*}{\sqrt{1+f_R}}, \quad (19)$$

$$\partial_\phi^2 V_{\text{eff}}(\phi, \rho^*) = \frac{d^2 V}{d\phi^2} + \beta^2 \kappa^2 e^{\beta \kappa \phi} \rho^* \quad (20)$$

$$= \frac{1}{3f_{RR}} - \frac{3R + 4f - Rf_R}{3(1 + f_R)^2} + \frac{\beta^2 \kappa^2 \rho^*}{\sqrt{1 + f_R}}. \quad (21)$$

In the solar system the deviation from GR must be small, i.e., $|f| \ll R$ and $|f_R| \ll 1$, which we will use for the approximation involved in the following derivation. Therefore,

$$\partial_\phi V_{\text{eff}} \simeq \frac{1}{\sqrt{6}\kappa} \left[(R - \kappa^2 \rho^*) + \left(2f - 3Rf_R + \frac{1}{2}f_R \kappa^2 \rho^* \right) \right], \quad (22)$$

$$\partial_\phi^2 V_{\text{eff}} \simeq \left(\frac{1}{3f_{RR}} - R + \frac{1}{6}\kappa^2 \rho^* \right) + \left(-\frac{4}{3}f + \frac{7}{3}Rf_R - \frac{1}{12}f_R \kappa^2 \rho^* \right). \quad (23)$$

The location of the potential minimum, ϕ_m , is given by $\partial_\phi V_{\text{eff}}(\phi_m, \rho^*) = 0$, and therefore satisfies

$$[R + 2f - 5Rf_R/2]_{\phi_m} \simeq \kappa^2 \rho^*, \quad (24)$$

or, to the lowest order,

$$R(\phi_m) \simeq \kappa^2 \rho^*. \quad (25)$$

Thus,

$$\phi_m \simeq -\frac{1}{2\beta\kappa} \ln [1 + f_R(R \simeq \kappa^2 \rho^*)] \quad (26)$$

$$\simeq -\frac{1}{2\beta\kappa} f_R(R \simeq \kappa^2 \rho^*). \quad (27)$$

We then obtain the formula for the mass,

$$m^2(\rho^*) = \partial_\phi^2 V_{\text{eff}}(\phi_m, \rho^*) \simeq \left[\left(\frac{1}{3f_{RR}} - \frac{5}{6}R \right) + \left(-f + \frac{11}{6}Rf_R \right) \right]_{R \simeq \kappa^2 \rho^*}, \quad (28)$$

and the relation between the thin-shell parameter and $f(R)$,

$$\frac{\Delta r_s}{r_s} \simeq -\frac{f_R(\kappa^2 \rho_{\text{out}}^*) - f_R(\kappa^2 \rho_{\text{in}}^*)}{2\Phi_s}. \quad (29)$$

In many viable $f(R)$ models, $|f_R(R_1)| \gg |f_R(R_2)|$ when $R_1 \ll R_2$, in order to fit the cosmic microwave background (CMB) and the big-bang nucleosynthesis (BBN) observational results that require the deviation from GR be tiny at early times, meanwhile generating the cosmic acceleration with a significant deviation from GR at late times. In this case,

$$\Delta r_s/r_s \simeq -f_R(\kappa^2 \rho_{\text{out}}^*)/2\Phi_s \quad \text{when} \quad \rho_{\text{out}}^* \ll \rho_{\text{in}}^*. \quad (30)$$

The condition $\rho_{\text{out}}^* \ll \rho_{\text{in}}^*$ is satisfied in many solar-system experiments.

IV. SOLAR-SYSTEM CONSTRAINTS ON $f(R)$ GRAVITY

To fit the solar-system constraints, the thin-shell condition in Eq. (17) needs to be satisfied in the solar system. This condition requires (1) the existence of the minimum of the effective potential $V_{\text{eff}}(\phi, \rho^*)$ and (2) the smallness of $|f_R|$.

From Eq. (28), the existence of the minimum of V_{eff} , i.e. $m^2 > 0$, entails (to the lowest order)

$$0 < R f_{RR} < 2/5 \quad \text{when} \quad R \simeq \kappa^2 \rho^* \gtrsim 3 \times 10^5 H_0^2, \quad (31)$$

where we have considered various environments in the solar system, with the mass density ρ^* ranging from 10^{-24} g/cm³ (Space) to 5.5 g/cm³ (Earth) and accordingly with R/H_0^2 ranging from 3×10^5 to $\mathcal{O}(10^{30})$. For simplicity we use $R \gtrsim 3 \times 10^5 H_0^2$ in the above expression.

Regarding the thin-shell condition, an upper bound $(\Delta r_s/r_s)_{\text{max}}$ of the thin-shell parameter constrains f_R via Eqs. (17) and (29) as follows.

$$0 < f_R(\kappa^2 \rho_{\text{in}}^*) - f_R(\kappa^2 \rho_{\text{out}}^*) < 2\Phi_s \cdot (\Delta r_s/r_s)_{\text{max}} \ll \Phi_s, \quad (32)$$

which gives an upper bound to the f_R variation. In the cases where

$$|f_R(\kappa^2 \rho_{\text{in}}^*)| \ll |f_R(\kappa^2 \rho_{\text{out}}^*)| \quad \text{for} \quad \rho_{\text{in}}^* \gg \rho_{\text{out}}^*, \quad (33)$$

it gives an upper bound to $|f_R(\kappa^2 \rho_{\text{out}}^*)|$:

$$0 < -f_R < 2\Phi_s \cdot (\Delta r_s/r_s)_{\text{max}} \ll \Phi_s \quad \text{when} \quad R \simeq \kappa^2 \rho_{\text{out}}^*. \quad (34)$$

In the following we will consider the upper bound $(\Delta r_s/r_s)_{\text{max}}$ obtained from the solar-system experiments involving the Sun and the Earth, respectively.

The experimental tests of the post Newtonian parameters in the solar system [31] give [27]

$$(\Delta r_s/r_s)_{\text{max}} = 1.15 \times 10^{-5}. \quad (35)$$

With $\Phi_\odot \simeq 2.12 \times 10^{-6}$ for the Sun and $\rho_{\text{out}}^* \simeq \rho_G^* \simeq 10^{-24}$ g/cm³, we obtain

$$-5 \times 10^{-11} < f_R < 0 \quad \text{when} \quad R/H_0^2 \sim 3 \times 10^5. \quad (36)$$

The case of the Earth experiments is more complicated. In this case the spherically symmetric system has three regions: the Earth, the atmosphere, and the beyond, with the

mass densities as follows.

$$\rho^*(r) = \begin{cases} \rho_\oplus^* \simeq 5.5 \text{ g/cm}^3 & \text{for } 0 < r < r_\oplus, \\ \rho_{\text{atm}}^* \simeq 10^{-3} \text{ g/cm}^3 & \text{for } r_\oplus < r < r_{\text{atm}}, \\ \rho_G^* \simeq 10^{-24} \text{ g/cm}^3 & \text{for } r > r_{\text{atm}}, \end{cases} \quad (37)$$

where the Earth radius $r_\oplus \simeq 6.4 \times 10^3 \text{ km}$ and the thickness of the atmosphere $\Delta d_{\text{atm}} = r_{\text{atm}} - r_\oplus \simeq 10\text{--}100 \text{ km}$. With the following thin-shell condition satisfied,

$$\frac{\Delta r_{\text{atm}}}{r_{\text{atm}}} \equiv \frac{\kappa(\phi_G - \phi_{\text{atm}})}{6\beta\Phi_{\text{atm}}} \ll 1, \quad \Phi_{\text{atm}} \equiv \frac{1}{6}\kappa^2\rho_{\text{atm}}^*r_{\text{atm}}^2, \quad (38)$$

the scalar field profile is [30]:

$$\phi(r) \simeq \begin{cases} \phi_\oplus & \text{for } 0 < r < r_\oplus, \\ \phi_{\text{atm}} & \text{for } r_\oplus < r < r_{\text{atm}}, \\ -\left(\frac{\beta\kappa}{4\pi}\right)\left(\frac{3\Delta r_\oplus}{r_\oplus}\right)\frac{M_\oplus e^{-m_G(r-r_{\text{atm}})}}{r} + \phi_G & \text{for } r > r_{\text{atm}}, \end{cases} \quad (39)$$

$$\frac{\Delta r_\oplus}{r_\oplus} \equiv \frac{\kappa(\phi_G - \phi_{\text{atm}})}{6\beta\Phi_\oplus} \ll 1, \quad \Phi_\oplus \equiv \frac{1}{6}\kappa^2\rho_\oplus^*r_\oplus^2, \quad (40)$$

where ϕ_\oplus , ϕ_{atm} and ϕ_G respectively denote the locations of the effective potential V_{eff} minima in the three regions, M_\oplus the mass of the Earth, and Φ_\oplus and Φ_{atm} the Newtonian potentials: $M_\oplus \simeq 6 \times 10^{24} \text{ kg}$, $\Phi_\oplus \simeq 7 \times 10^{-10}$ and $\Phi_{\text{atm}} \simeq 10^{-13}$. Note that the thin-shell condition in Eq. (38) automatically leads to $\Delta r_\oplus/r_\oplus \ll 1$. The experimental bounds of the thin-shell parameters, $0 < \Delta r_i/r_i < (\Delta r_i/r_i)_{\text{max}} \ll 1$ for $i = \oplus, \text{atm}$, then give upper bounds to $|f_R|$:

$$0 < -f_R(\kappa^2\rho_G^*) < 2\Phi_i \cdot (\Delta r_i/r_i)_{\text{max}} \ll \Phi_i, \quad i = \oplus, \text{atm}, \quad (41)$$

in the case where $|f_R(\kappa^2\rho_{\text{atm}}^*)| \ll |f_R(\kappa^2\rho_G^*)|$ for $\rho_{\text{atm}}^* \gg \rho_G^*$.

An essential experimental bound comes from the basic requirement that the atmosphere has a thin shell, i.e., the thickness of the thin shell should be smaller than that of the atmosphere: $0 < \Delta r_{\text{atm}}/r_{\text{atm}} < \Delta d_{\text{atm}}/r_{\text{atm}}$. Taking the thickness of the atmosphere $\Delta d_{\text{atm}} \simeq 50 \text{ km}$, i.e. $\Delta d_{\text{atm}}/r_{\text{atm}} \simeq 8 \times 10^{-3}$, we obtain a very stringent bound to f_R :

$$-10^{-15} \lesssim f_R < 0 \quad \text{when} \quad R/H_0^2 \sim 3 \times 10^5. \quad (42)$$

To sum up, from the thin-shell condition in the solar system we have obtained a constraint on Rf_{RR} in Eq. (31) and a stringent upper bound to $|f_R|$ in Eq. (42) for non-constant $f(R)$.³

³ Note that f_R and f_{RR} can vanish in the case of constant $f(R)$ that is equivalent to the Λ CDM model and therefore not taken into consideration when we investigate $f(R)$ gravity for possible deviations from GR.

They respectively come from the requirement of the existence of the effective potential $V_{\text{eff}}(\phi, \rho^*)$ minimum in the solar system and from the thin-shell condition in the atmosphere, with the precondition that $|f| \ll R$ and $|f_R| \ll 1$ in the solar system, and $|f_R(\kappa^2 \rho_{\text{atm}}^*)| \ll |f_R(\kappa^2 \rho_G^*)|$.

The constraints on $f(R)$ we obtained can be conveniently applied to the $f(R)$ models where the functional forms of $f(R)$ are given. For demonstration, here we apply the constraint on f_R in Eq. (42) to the following widely considered models [20]:

$$f(R) = -\lambda R_c f_1(x), \quad x \equiv R/R_c, \quad \lambda, R_c > 0; \quad (43)$$

- (1) $f_1(x) = x^p, 0 < p < 1$;
- (2) Hu and Sawicki [7]: $f_1(x) = x^{2n}/(x^{2n} + 1), n > 0$;
- (3) Starobinsky [8]: $f_1(x) = 1 - (1 + x^2)^{-n}, n > 0$;
- (4) Tsujikawa [10]: $f_1(x) = \tanh(x)$;
- (5) Linder [13]: $f_1(x) = 1 - e^{-x}$.

Considering $\lambda \sim \mathcal{O}(1)$, $R_c \sim \kappa^2 \rho_c$ (where ρ_c is the critical density at the present time) and $R/R_c \sim 10^5$, we find that the constraint in Eq. (42) requires $p < 10^{-10}$ in Model (1) and $n > 1$ in Models (2) and (3), and it is well satisfied in Models (4) and (5). In addition, the constraint on Rf_{RR} in Eq. (31) is also satisfied in these models. Note that in these models and under the above consideration, the conditions $|f| \ll R$, $|f_R| \ll 1$ and $|f_R(\kappa^2 \rho_{\text{atm}}^*)| \ll |f_R(\kappa^2 \rho_G^*)|$ are satisfied, so that we can legitimately use the simple and stringent constraint on $f_R(\kappa^2 \rho_G^*)$, i.e. the constraint in Eq. (42).

V. CONCLUSION

The cosmological tests and the local tests give essential constraints on the deviation from GR in $f(R)$ gravity at different values of the Ricci scalar R , which correspond to the constraints at the cosmological scales at different epochs of the cosmic expansion history. Roughly speaking, CMB and BBN stringently constrain the large- R (early-time) behavior of $f(R)$, the solar-system tests stringently constrain the moderate- R (middle-age) behavior,

and the cosmological observations about the late-time universe constrain the small- R (late-time) behavior.

With regard of the cosmic history, the constraints on the $f(R)$ modified gravity at the cosmological scales from the early times to the present are summarized as follows.

- $z \gtrsim 10^3$: The deviation from GR, such as f_R , should be small at early times when the redshift $z \gtrsim 10^3$, as required by the CMB and the BBN observations.
- $z \gtrsim 70$: $0 < Rf_{RR} < 2/5$ when $R/H_0^2 \gtrsim 3 \times 10^5$ that roughly corresponds to the epoch $z \gtrsim 70$. This is a basic requirement in the chameleon mechanism for the solar system.
- $z \sim 70$: $-10^{-15} < f_R < 0$ when $R/H_0^2 \sim 3 \times 10^5$ that roughly corresponds to the time when $z \sim 70$. We obtain this stringent constraint from the thin-shell condition required in the solar-system test, particularly the test in the atmosphere.
- $z \sim \mathcal{O}(1)$: The deviation from GR needs to be significant in order to explain the cosmic acceleration at the present epoch.

According to the above constraints, in the viable $f(R)$ models of the late-time cosmic acceleration the deviation from GR should be small when $z \gtrsim 70$ but become significant at the recent epoch. The above constraints give simple, clear requirements one can conveniently utilize to examine the viability of the $f(R)$ models with various functional forms of $f(R)$.

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